Options on Quantum Money: Quantum Path- Integral With Serial Shocks

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***Abstract* — The author previously developed a numerical multivariate path-integral algorithm, PATHINT, which has been applied to several classical physics systems, including statistical mechanics of neocortical interactions, options in financial markets, and other nonlinear systems including chaotic systems. A new quantum version, qPATHINT, has the ability to take into account nonlinear and time-dependent modifications of an evolving system. qPATHINT is shown to be useful to study some aspects of serial changes to systems. Applications to options on quantum money and blockchains in financial markets are discussed.**

***Keywords* — path integral, quantum mechanics, blockchains, parallel code, financial options, path integral**

I. INTRODUCTION

The author previously developed a numerical multivariate path-integral algorithm, PATHINT. PATHINT has been used for several systems [1][2][3][4][5], The second section briefly describes path integrals and the numerical PATHINT algorithm.

A quantum version, qPATHINT, has the ability to take into account nonlinear and serial time-dependent modifications of an evolving system. Quantum computing is here, and in the near future it will be applied to financial products, some using blockchain technologies. It not implausible to assume that soon there will be derivatives developed on these products, e.g., options. Then, similar to cases in classical real spaces with PATHINT, qPATHINT is now poised to calculate derivatives in quantum complex spaces. qPATHINT has been successfully baselined to PATHINT. qPATHINT goes beyond simply using quantum computation of derivatives, since the space of the dependent variables themselves may live in quantum worlds. Another paper has applied qPATHINT to a neuroscience problem [6].

In the second section, qPATHINT is described in the context of quantum money (QM, not to be confused here with the more common use of "QM" for quantum mechanics). Previous papers have addressed the use of PATHINT to qPATHINT versus the similar generalization of another numerical path-integral algorithm, PATHTREE to qPATHTREE [7][8].

The third section describes the PATHINT and qPATHINT algorithms.

The fourth section describes the use of qPATHINT and PATHINT to compare options calculations in the presence of serial shocks.

The Conclusion stresses that these applications of qPATHINT give proofs of concept of these new algorithms and codes.

II. QUANTUM MONEY

Since the early 1970's there have been occasional papers proposing QM [9][10][11][12][13][14][15][16]. There are good reasons to consider QM, including possibilities of counterfeit-proof currency, and combining of such currency with blockchain technologies yielding improved efficiencies of mining and permitting scaling beyond today's blockchains.As yet, there is not a clear proposal for just how QM would be implemented or exchanged with classical money. However, quantum computing is here now and rapidly growing [17], which will be applied in many ways to current financial markets. It seems reasonable that soon financial markets will be expanded to include quantum variables, and financial markets will determine how QM is to be valued and how it may ne exchanged with current financial instruments.

Note that this paper does not address the problems in defining QM. This study does address how options on such QM can be calculated.

2.1 QUANTUM OPTIONS ON QUANTUM MARKETS

It seems that "if" QM is not much of an issue. When QM does arrive, it is clear that options on quantum markets will be required for purposes of hedging and speculation. Quantum options on quantum markets will require technologies similar to those required by trading options on classical financial markets. For example, American options, that may be exercised before maturation, is a key technology in today's markets requiring numerical algorithms. Furthermore, similar to today's technologies, probability distributions of prices in real markets will not generally be simple Gaussian or log-normal distributions that yield closed form options solutions. Real-world data, especially given seasonal changes and taxation issues, require fits to determine actual distributions.

2.2 PATH-INTEGRAL METHODS APPLIED TO MARKETS

The author has developed two sets of path-integral related algorithms, PATHINT [1][2][3][4][5] and PATHTREE [8] These algorithms have been applied to several disciplines, including financial options [3][18]. Other authors also have applied classical path-integral techniques to options [19].

A key feature of these algorithms is capabilities of developing the evolution of multivariate probability distributions with quite generally nonlinear means and (co-)variances and time dependencies. This is important to take into account known future sudden changes in markets, e.g., dividends, as well as possible drastic crashes and mini-crashes which at least should be regularly included in risk analyses.

2.3. QPATHINT

qPATHINT is a code that is developed from PATHINT, useful for propagation of quantum wave functions [7][6]. This has been tested in a neuroscience project [6].

qPATHINT has been baselined to PATHINT, using real variables as input. This paper gives results of comparing PATHINT with qPATHINT, for a model of options that simply generalities classical variables x to complex variable $z = x + ix$, in the presence of random serial shocks. PATHINT and qPATHINT are particularly well suited to process serial shocks, much more so than other Monte Carlo methods that also are applied to classical and quantum development of probabilities and wave functions [20].

III. PATH INTEGRALS

The path-integral representation for the short-time propagator P is given in terms of a Lagrangian L by

$$P(S', t' = t + dt \mid S, t) = \frac{1}{2\pi g^2 \Delta t} \exp(-L dt)$$

$$L = \frac{(dS/dt - f)^2}{2g^2}$$

$$\frac{dS}{dt} = \frac{S' - S}{dt} \tag{1}$$

The path integral defines the long-time evolution of P in terms of the action A,

$$P[S(t)]dS(t) = \int \ldots \int DS \exp(-A)$$

$$A = \int_{t_0}^{t} dt' L$$

$$L = \Lambda \Omega^{-1} \int d\vec{S} L$$

$$DS = \prod_{s=1}^{u+1} \prod_{\nu=1}^{\Lambda} (2\pi \, dt \, g_s^{\nu\,2})^{-1/2} dS_s^{\nu} \delta[S_t = S(t)][\delta[S_0 = S(t_0)] \tag{2}$$

where ν labels the $\vec{S}$-space over the volume Ω, and s labels the $u+1$ time intervals, each of duration dt, spanning $(t - t_0)$. The path integral is a faithful mathematical representation of (properly defined) Fokker-Planck partial differential equations and Langevin stochastic differential equations [21].

3.1 PATHINT

qPATHINT is motivated by a previous non-Monte-Carlo multivariable generalization of a numerical path-integral algorithm [22][23][24], PATHINT, used to develop the long-time evolution of the short-time probability distribution as applied to several studies in chaotic systems [2][5], neuroscience [1][2][4], and financial markets, including two-variable volatility of volatility [3].

These studies suggested applications of some aspects of this algorithm to the standard binomial tree, PATHTREE [8], which is in development for a quantum version, qPATHTREE. However, as noted previously [7], [q]PATHINT quite easily extends beyond a diagonal across a (multivariate) drift, enabling oscillatory distributions/wave-functions to be calculated, whereas this is not very practical for [q]PATHTREE.

PATHINT develops bins T_{ij} about diagonal terms in the Lagrangian.

$$P_i(t+\Delta t) = T_{ij}(\Delta t) P_j(t)$$

$$T_{ij}(\Delta t) = \frac{2}{\Delta S_{i-1} + \Delta S_i} \int_{S_i - \Delta S_{i-1}/2}^{S_i + \Delta S_i/2} dS \int_{S_j - \Delta S_{j-1}/2}^{S_j + \Delta S_j/2} dS' G(S, S'; \Delta t) \quad (3)$$

T_{ij} is a banded matrix representing the Gaussian nature of the short-time probability centered about the (varying) drift.

Fitting data with the short-time probability distribution, effectively using an integral over this epoch, permits the use of coarser meshes than the corresponding stochastic differential equation. The coarser resolution is appropriate, typically required, for numerical solution of the time-dependent path-integral. By considering the contributions to the first and second moments of ΔS^G for small time slices θ, conditions on the time and variable meshes can be derived [22]. The time slice essentially is determined by $\theta \le \bar{L}^{-1}$, where $\bar{L}$ is the "static" Lagrangian with $dS^G / dt = 0$, throughout the ranges of S^G giving the most important contributions to the probability distribution P. The variable mesh, a function of S^G, is optimally chosen such that ΔS^G is measured by the covariance $g^{GG'}$, or $\Delta S^G \approx (g^{GG} \theta)^{1/2}$.

PATHINT was generalized by the author to process arbitrary N variable spaces, but in practice only N = 2 was used because of intensive computer resources. The author was Principal Investigator, National Science Foundation (NSF) Pittsburgh Supercomputing Center (PSC) Grant DMS940009P during 1994-1995 on a project, "Porting Adaptive Simulated Annealing and Path Integral Calculations to the Cray; Parallelizing ASA and PATHINT Project (PAPP)". Eight volunteers were selected from many applicants and the PATHINT code was seeded to work on parallel machines. No further work has been done since that time to further develop parallel-coded applications.

3.2 qPATHINT

Similar to the development of qPATHINT described above, the PATHINT C code of about 7500 lines of code was rewritten for the GCC C-compiler to use complex double variables instead of double variables.

qPATHINT, using real variables, was baselined to PATHINT by obtaining numerical agreement to previous results using PATHINT to calculate options for financial markets [3]. Note that options calculations include calculations of evolving probability distributions, making such codes very useful for similar calculations in other disciplines. qPATHINT evolves a wave function whose absolute square at any node is a probability, e.g., to determine payoffs at nodes when calculating American options.

3.3 Serial Shocks

This path-integral study is used to examine the nature of disturbances on the propagation of the wave function ψ due to serial shocks. The standard C-code uniform integer random number generator, rand(), is scaled to develop random real numbers within [-1,1], which adds a multiplier η to the drift, e.g., drift = (1 + Rand) drift.

IV. COMPARISON OF PATHINT AND QPATHINT

4.1 Probability Distributions

Table 1 gives a sample of the values of coordinates and distributions from PATHINT, resp., with $\eta = 0$ (no shocks), and a sample of the values of distributions from PATHINT, resp., with $\eta = 0.5$.

Table 2 gives a sample of the values of coordinates and real and imaginary distributions from qPATHINT, resp., with $\eta = 0$ (no shocks), and a sample of the values of coordinates and real and imaginary distributions from qPATHINT, resp., with $\eta = 0.5$.

width=0.8tw

TABLE 1: COLUMNS 1 AND 2 HAVE VALUES OF COORDINATES AND DISTRIBUTIONS FROM PATHINT, RESP., WITH A SHOCK MULTIPLIER OF 0. COLUMNS 3 HAS VALUES OF DISTRIBUTIONS FROM PATHINT, WITH A SHOCK MULTIPLIER OF 0.5.

x-P	P-0	P-0.5
.12942	6.25793	6.2521
.30794	6.50949	6.50477
.49967	6.59601	6.59499
.70463	6.55281	6.55491
.92282	6.42107	6.42265
.15425	6.2352	6.23592
.39892	6.01785	6.01706
.65683	5.7812	5.77962
.928	5.53064	5.52912
.21243	5.26809	5.26886
.51011	4.9941	4.99533
.82106	4.7087	4.71042
.14528	4.41184	4.41369
.48276	4.10344	4.10557
.83352	3.78344	3.78544
.19755	3.45182	3.45277
.57485	3.10865	3.10997
.96544	2.75435	2.75566
.3693	2.39016	2.39118
.78645	2.01916	2.01932
.21688	1.64766	1.64762
.66059	1.28647	1.28589
.11759	0.950734	0.949436
.58788	0.657414	0.656073
.07146	0.420468	0.419593
.56833	0.246061	0.245715

width=1.0tw

TABLE 2: COLUMNS 1, 2 AND 3 HAVE VALUES OF COORDINATES AND REAL AND IMAGINARY DISTRIBUTIONS FROM QPATHINT, RESP., WITH A SHOCK MULTIPLIER OF 0. COLUMNS 4 AND 5 HAVE VALUES REAL AND IMAGINARY DISTRIBUTIONS FROM QPATHINT, RESP., WITH A SHOCK MULTIPLIER OF 0.5.

z	P-rl	P-im	qP-rl	qP-im
.08059+i1.08059	1.26001	1.22204	1.25961	1.22156
.18684+i1.18684	1.26603	1.23262	1.26553	1.23204
.2981+i1.2981	1.23223	1.20642	1.2316	1.20573
.41437+i1.41437	1.16609	1.14872	1.16505	1.14771
.53566+i1.53566	1.07644	1.06651	1.07553	1.06563
.66197+i1.66197	0.971171	0.966859	0.970277	0.965963
.79328+i1.79328	0.856197	0.855897	0.855609	0.855274
.92962+i1.92962	0.73569	.738588	0.735898	0.738703
.07097+i2.07097	.613086	0.619194	0.613698	0.61973
.21735+i2.21735	0.492107	0.501867	0.492577	0.502338
.36874+i2.36874	0.377303	0.390973	0.377838	0.391508
.52515+i2.52515	0.273802	0.290938	0.274335	0.291488
.68658+i2.68658	0.186369	0.205707	0.18692	0.206299
.85303+i2.85303	0.118264	0.138147	0.118666	0.13859
.02451+i3.02451	0.0706271	0.0897805	0.0708143	0.089998
.201+i3.201	0.042876	0.0610806	0.0429495	0.0611828
.38252+i3.38252	0.0337658	0.0518205	0.033812	0.051894

4.2. Options

Options models describe the market value of an option, V as

$$V = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{4}$$

where S is the asset price, and σ is the standard deviation, or volatility of S, and r is the short-term interest rate. For example, the basic equation can apply to a number of one-dimensional models of interpretations of prices given to V, e.g., puts or calls, and to S, e.g., stocks or futures, dividends, etc.

The basic options model considers a portfolio Π in terms of Δ,

$$\Pi = V - \Delta S \tag{5}$$

in a market with Gaussian-Markovian ("white") noise X and drift μ,

$$\frac{dS}{S} = \sigma dX + \mu dt \tag{6}$$

where $V(S,t)$ inherits a random process from S,

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt \tag{7}$$

This yields

$$d\Pi = \sigma \left(\frac{\partial V}{\partial S} - \Delta \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt \tag{8}$$

Which defines the "Greeks" as

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

$$\Theta = \frac{\partial \Pi}{\partial t}$$

$$\Upsilon = \frac{\partial \Pi}{\partial \sigma}$$

$$\rho = \frac{\partial \Pi}{\partial r} \tag{9}$$

The portfolio can be "risk-neutral," if Δ is chosen such that

$$\Delta = \frac{\partial V}{\partial S} \tag{10}$$

The expected risk-neutral return of Π is

$$d\Pi = r\Pi dt = r(V - \Delta S)dt \tag{11}$$

4.3 Options Calculations

Table 3 gives path-integral (PI) options values from PATHINT and qPATHINT (here, real parts only), for American (early exercise) and European call and put (AC, AP, EC, EP) calculations. Columns 2 and 3 are for PATHINT and qPATHINT, resp., with $\eta = 0$. Columns 4 and 5 are for PATHINT and qPATHINT, resp., with $\eta = 0$. A value of 9 off-diagonal terms are used on each side of the diagonal kernel. The model uses a noise of S^x, where S is the underlying price and x is an exponent. The underlying price is taken to be 7.0. A strike value of 7.5 is used for this table. The risk-free rate is taken to be 0.0675. The cost of carry is taken to be 0. A daily volatility of 0.00793725 is used, and this parameter is taken to be real for both PATHINT and qPATHINT.

There is no additional drift added, but a drift arises from the nonlinear noise used [3][18]. In this context, note that shocks can affect Greeks with "p" in Table 3 quite severely, where "p" denotes an additional order of derivatives, e.g., VegapPI (second derivative of Υ with respect to volatility) is very sensitive to shocks in this particular drift as described above in the section Serial Shocks.
width=1.0tw

Table 3: Columns 1 designates path-integral (PI) options values from PATHINT and qPATHINT (here, real parts only), for American and European call and put (AC, AP, EC, EP) calculations. Columns 2 and 3 are for PATHINT and qPATHINT, resp., with a shock multiplier of 0. Columns 4 and 5 are for PATHINT and qPATHINT, resp., with a shock multiplier of 0.5.

PI	P-0	qP-0	P-0.5	qP-0.5
PricePIEC	0.288756	0.288756	0.208977	0.208977
DeltaPIEC	0.380897	0.380897	0.303623	0.303623
ThetaPIEC	-0.109478	-0.109478	-0.105974	-0.105974
VegaPIEC	3.02971	3.02977	2.95609	2.96872
RhoPIEC	-0.312973	-0.313763	-0.425339	-0.426127
GammaPIEC	0.324376	0.162188	0.300693	0.150346
DeltapPIEC	0.755352	0.755352	1.39363	1.37307
VegapPIEC	1.17653	1.17745	51.1558	51.3351
PricePIEP	0.515915	0.515915	0.663187	0.663187
DeltaPIEP	-0.518676	-0.518676	-0.596342	-0.596342
ThetaPIEP	-0.095225	-0.095225	-0.0768704	-0.0768704
VegaPIEP	3.0557	3.05573	2.75136	2.74061
RhoPIEP	-0.991903	-0.994407	-0.901005	-0.903511
GammaPIEP	0.324985	0.162493	0.306842	0.153421
DeltapPIEP	0.718208	0.718208	1.19527	1.20402
VegapPIEP	1.83548	1.83482	72.1266	72.712
PricePIAC	0.295256	0.295256	0.212971	0.212971
DeltaPIAC	0.39382	0.39382	0.312081	0.312081
ThetaPIAC	-0.117647	-0.117647	-0.111751	-0.111751
VegaPIAC	3.10626	3.10621	3.0099	3.02144
RhoPIAC	-0.244397	-0.24536	-0.340059	-0.341156
GammaPIAC	0.346264	0.173132	0.316041	0.15802
DeltapPIAC	0.805888	0.805551	1.44574	1.42869
VegapPIAC	1.54957	1.70147	40.5332	40.5649
PricePIAP	0.531893	0.531893	0.687709	0.687709
DeltaPIAP	-0.545672	-0.545672	-0.635442	-0.635442
ThetaPIAP	-0.110369	-0.110369	-0.0972145	-0.0972145
VegaPIAP	3.11602	3.11596	2.7026	2.68574
RhoPIAP	-0.555006	-0.559152	-0.537712	-0.541252
GammaPIAP	0.365988	0.182994	0.362628	0.181314
DeltapPIAP	0.823279	0.823017	1.5532	1.53829
VegapPIAP	1.54039	1.66711	-42.8045	-42.4406

V. CONCLUSION

A numerical path-integral algorithm, PATHINT, has been generalized to complex variable spaces, resulting in a new qPATHINT code useful for quantum wave functions and/or quantum probability functions.

PATHINT has already applied to various systems, including financial-market options, chaotic studies, and neuroscience. Similar to PATHINT, qPATHINT's accuracy is best for moderate-noise systems. Much CPU in qPATHINT is used just calculating the distribution at all nodes. The added generalization of dealing with N dimensions in qPATHINT requires a lot of overhead taking care of indices and boundaries within many for-loops. Parallel processing makes these codes more efficient in real-time. qPATHINT is applied to options on QM. A proof of principle has been demonstrated, which is poised to handle quantum options on quantum systems, in finance and in blockchains. qPATHINT can be useful in multiple disciplines, e.g., neuroscience and financial markets. A neuroscience example is given here. A proper treatment of financial options often requires inclusion of aperiodic dividends and distributions that deviate nonlinearly from Gaussian of log-normal [3][8].

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